(1) Let $x_1, x_2, \cdots, x_n$ be a random sample from a distribution having the p.d.f.

$$f(x;\theta) = \frac{\{x(1-x)\}^{\theta-1}}{B(\theta,\theta)} \; ; \; 0<x<1, \theta>0$$

$B(1,1) = \frac{\Gamma1\,\Gamma1}{\Gamma2} = 1$

Test

$H_0 : \theta = 1$

ag. $H_1 : \theta = 2$

Likelihood function is $L(\theta, \underline{x}) = \frac{\prod_{i=1}^{n} x_i^{\theta-1} (1-x_i)^{\theta-1}}{\{B(\theta,\theta)\}^n}$

$$L_{H_0}(\theta=1, x) = \frac{\prod_{i=1}^{n} x_i^{1-1} (1-x_i)^{1-1}}{\{B(1,1)\}^n} = \frac{1}{\{B(1,1)\}^n} = 1.$$

$$L_{H_1}(\theta=2, x) = \frac{\prod_{i=1}^{n} x_i^{2-1} (1-x_i)^{2-1}}{\{B(2,2)\}^n}$$

$B(2,2) = \frac{1 \cdot 1}{\Gamma4} = \frac{1}{6}$

$$\therefore \frac{L_{H_1}}{L_{H_0}} = \frac{\{B(2,2)\}^n}{\{B(1,1)\}^n} \prod_{i=1}^{n} x_i(1-x_i) = \frac{1}{6^n} \prod_{i=1}^{n} x_i^n (1-x_i)^n$$

The MP critical region. $W : \left\{ \underline{x} ; \frac{\prod x_i^n (1-x_i)^n}{6^n} > k \right\}$

$\equiv W : \left\{ \underline{x} : \prod x_i^n (1-x_i)^n > k' \right\}$

$\equiv W : \left\{ \underline{x} : \sum n \log x_i + n \sum \log(1-x_i) > k' \right\}$

$\equiv W : \left\{ \underline{x} : \sum \log x_i + \sum \log(1-x_i) > c^0 \right\}$

$\boxed{\equiv W : \left\{ \underline{x} : \prod x_i (1-x_i) > c \right\}}$ $\longrightarrow$ Final Answer.

Q.2. MP Test

$$H_0: f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty$$

$$H_1: f(x) = \frac{1}{2} e^{-|x|}, \quad -\infty < x < \infty$$

(single observation)

Answer

$$W = \left\{ \underline{x} : \frac{\frac{1}{2} e^{-|x|}}{\frac{1}{\sqrt{2\pi}} e^{-x^2/2}} > k \right\}$$ where k is positive constant to be determined by the size condition.

$$\equiv W = \left\{ \underline{x} : e^{-|x| + |x|^2/2} > k' \right\} \qquad \left[\text{see } x^2 = |x|^2 \right]$$

$$= \left\{ \underline{x} : -|x| + |x|^2/2 > \log k' \right\}$$

$$= \left\{ \underline{x} : -|x| + |x|^2/2 > k'' \right\}$$

$$= \left\{ \underline{x} : -2|x| + |x|^2 > k''' \right\}$$

$$\doteq \left\{ \underline{x} : \text{left side is a quadratic form in } |x| \right\}$$

$$= \left\{ \underline{x} : -2|x| + |x|^2 + 1 > k''' + 1 \right\}$$

$$= \left\{ \underline{x} : (|x| - 1)^2 > k'''' \right\} \longrightarrow \text{a function of } |x|.$$

$$= \left\{ \underline{x} : |x| > c \text{ or } |x| < c \right\}$$

Note that, k', k'', k''', k'''', c are all suitably chosen constants.

$\boxed{\text{MP critical region is } |x| > c \text{ or } |x| < c}$.

(3) Look at this problem, this is a special problem as the distribution of x depends on the parameter θ. Neyman-Pearson fundamental lemma fails to construct MP region for such example if there are n sample observations $x_1, x_2, \cdots, x_n$.

In fact, for a single observation based test construction the critical region might be set as

$$W : \{x : X > a\}$$

a is suitably chosen constant formed from size condition

$$P_{H_0}\{X > a\} = \alpha.$$

Remember For this problem answer will be MP test is $W: X > a$ where a can be ~~construct~~ determined from size condition.

$$P_{H_0}\{X > a\} = \alpha.$$

4) Examine whether a test critical region exists for testing $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$ for the parameter θ of the distribution

$$f(x;\theta) = \frac{1+\theta}{(x+\theta)^2}, \quad 1 \le x < \infty$$

(Based on one observation)

Solution By N-P lemma

$$W = \left\{ x : \frac{L(x,\theta_1)}{L(x,\theta_0)} = \frac{1+\theta_1}{(x+\theta_1)^2} \times \frac{(x+\theta_0)^2}{1+\theta_0} > k \right\}$$

$$= \left\{ x : \frac{(x+\theta_0)^2}{(x+\theta_1)^2} > k'' \right\}$$

The left hand side can not be put in the form of sample observation only, therefore not satisfying the condition of N-P lemma.

No MP critical region exists.

5) P = Probability that a given die shows even number. $H_0: p = \frac{1}{2}$ against $H_1: p = 1/3$. Toss the die twice and accept H_0 if both times it shows even number. Prob{type I error} and Prob{type II error} ??

X: number of even points in two tosses of a die

$X \sim$ Binomial$(2, p)$.

$W = \{X : 0, 1\}$, $A = \{X : 2\}$.

$$\text{Prob}\{\text{type I error}\} = P(X \in W / H_0) = P(X = 0 \text{ or } 1 / p = \tfrac{1}{2})$$

$$= \binom{2}{0}\left(\tfrac{1}{2}\right)^0\left(\tfrac{1}{2}\right)^2 + \binom{2}{1}\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^1$$

$$= \tfrac{1}{4} + \tfrac{1}{2} = \tfrac{3}{4}$$

$$\text{Prob.}\{\text{type II error}\} = P(X = 2 / H_1)$$

$$= \binom{2}{2}\left(\tfrac{1}{3}\right)^2\left(\tfrac{2}{3}\right)^0 = \tfrac{1}{9}.$$

6) Let X have p.d.f. $f(x) = \frac{1}{\theta} e^{-x/\theta}, 0 < x < \infty, \theta > 0$
$H_0: \theta = 2$ against $H_1: \theta = 1$. Use the random sample X_1 and X_2 and define $W = \{(x_1, x_2) : 9.5 \le x_1 + x_2\}$. Find the power and level of significance.

$X_1 \sim Exp(\theta)$ $>$ random sample
$X_2 \sim Exp(\theta)$ so independent.

Now $Y = X_1 + X_2 \sim Gamma(2, \theta)$

$$\text{Power} = P(Y \in W / \theta = 1)$$
$$= P(X_1 + X_2 \geq 9.5 / \theta = 1).$$
$$= \int_{9.5}^{\infty} \frac{1}{\theta^2} e^{-x/\theta} x^{2-1} dx / \theta = 1$$
$$= \int_{9.5}^{\infty} \frac{1}{1} e^{-x/1} \cdot x^1 dx = \int_{9.5}^{\infty} e^{-x} \cdot x \, dx.$$
$$= (9.5 + 1) e^{-9.5} = 10.5 e^{-9.5}$$